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A RAND NOTE

The Geometry of a Satellite–Ballistic Missile Engagement

Michael D. Miller

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This Note describes in mathematical terms the dynamic geometry between a constellation of satellites deployed for ballistic missile defense (BMD) and the missiles the satellites are engaging. Formulas, readily translatable into computer code, are given for such engagement parameters as slant range, closing velocity, and line-of-sight incidence angle in terms of satellite and missile position and velocity data. These formulas are the foundation of a model developed at Rand to support study of the BMD capability of various satellite armament concepts.

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The Geometry of a Satellite–Ballistic Missile Engagement

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PREFACE AND SUMMARY

This Note describes in mathematical terms the dynamic geometry between a constellation of satellites deployed for ballistic missile defense (BMD) and the missiles the satellites are engaging. Formulas, readily translatable into computer code, are given for such engagement parameters as slant range, closing velocity, and line-of-sight incidence angle in terms of satellite and missile position and velocity data. These formulas are the foundation of a model developed at The RAND Corporation to support study of the BMD capability of various satellite armament concepts.

This work was initiated at RAND under sponsorship of the Directed Energy Office of the Defense Advanced Research Projects Agency. Sponsorship was transferred to the Directed Energy Office of the Strategic Defense Initiative Organization when that agency was formed. The study was carried out in RAND's National Defense Research Institute, a Federally Funded Research and Development Center sponsored by the Office of the Secretary of Defense.

THE GEOMETRY OF A SATELLITE-BALLISTIC MISSILE ENGAGEMENT

Our objective is to describe the dynamic geometry of an engagement in which a constellation of satellites is defending against ground-based ballistic missile launches. The formulas presented allow one to keep track simultaneously of satellite motion, missile motion, and earth rotation, and thus permit the determination of such variables as slant range, closing velocity, line-of-sight visibility, and missile/satellite orientation.

ASSUMPTIONS

All satellites are assumed to be in circular orbits at a common altitude. We assume a spherical earth of uniform density and ignore possible orbit perturbations due to extraterrestrial bodies. The constellation consists of m orbit rings with n satellites per ring and is called an *$m \times n$ constellation*. All rings have the same inclination with respect to the equatorial plane, with satellites in a given ring equally spaced and traveling in the same direction. We consider two ring-spacing options:

> Option A: The ascending nodes of the rings* are equally spaced around the equator. In the case of polar orbits with m even, each ring is coincident with another ring, with satellites in the two rings traveling in opposite directions.
>
> Option B: The ascending nodes of the rings are equally spaced around one-half the equator.

These options are illustrated for two orbit inclinations in Fig. 1.

We select a fixed reference ring, called ring 1, and label the remaining rings 2, 3, 4, ..., according to the position of their ascending nodes westward from ring 1. This reference ring can be any of the

*That is, the point at which satellites in the ring cross the equatorial plane and ascend into the northern hemisphere.

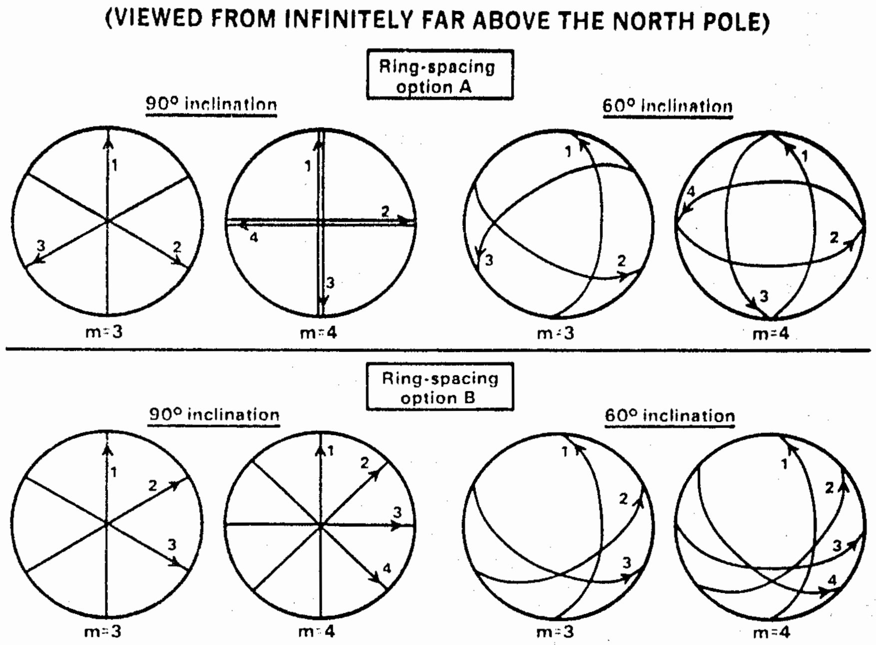

Fig. 1 — Examples of constellation configurations

m rings under option A; under option B, it is that ring whose ascending node is the easternmost along that half of the equator containing all m ascending nodes. Select some fixed satellite in ring 1 and agree that it occupies position 1 in this ring. This satellite, labeled 1-1, will henceforth be called the *reference satellite*. Label the positions of other satellites in ring 1 by 2, 3, 4, ..., moving forward (i.e., in the direction of satellite motion) from the reference satellite. In general, satellite i-j occupies position j in ring i. The positions of satellites in rings 2 through m, relative to those in ring 1, are determined by the *constellation phasing fraction* ρ ($0 \leq \rho < 1$). In particular, we assume that satellite 2-1 is $Q = 360\rho/n$ deg ahead (in argument of latitude) of satellite 1-1, satellite 3-1 is Q deg ahead of 2-1, and so on. For ring-spacing option A, in order that satellites in rings m and 1 occupy the same relative positions as those in other pairs of adjacent rings, ρ is restricted to the values 0, 1/m, 2/m, ..., (m - 1)/m. This requirement cannot be met under option B since satellites in rings m and 1 travel in opposite directions; thus ρ can assume any value between 0

and 1. So that we can assign times to various events, we assume that at some *0-hour* time, the reference satellite crosses the equator at E deg longitude heading north.

To facilitate our presentation, we introduce an earth-referenced (rotating) xyz coordinate system, with the origin (0,0,0) at the earth's center. The equator lies in the xy-plane with the positive x-axis passing through the Greenwich prime meridian, the positive y-axis lying 90 deg east, and the positive z-axis passing through the north pole. As is conventional, latitudes are measured northward from the equator, longitudes eastward from the positive x-axis, azimuths clockwise from due north, and path elevation angles upward from horizontal.

The ballistic missile (which we refer to as the ICBM) is sited at a particular latitude and longitude and is launched with a known azimuth relative to the above coordinate system. It is assumed that sufficient ICBM trajectory data are available (e.g., from a powered flight simulator) to establish ICBM position and velocity vectors at all times of interest. An example of such data is shown in Table 1 (p. 9) for the boost phase period of a hypothetical ICBM. Although, in general, six trajectory elements are needed to establish position and velocity, the actual six used can vary depending on the output of the model used to simulate the ICBM flight. Rand's model COMET (see R-3240-ARPA, forthcoming) outputs position and velocity directly. To accommodate models that do not directly output these data, we describe in the pages that follow how to calculate these vectors and other relevant parameters, assuming as basic inputs the following six ICBM trajectory elements:

- ICBM altitude
- ICBM latitude
- ICBM longitude
- ICBM speed
- ICBM path azimuth
- ICBM path elevation angle

The formulas incorporate the following definitions and conventions:

Definitions

Satellite orbit inclination (θ)--the angle between the equatorial plane and the plane of the satellite's orbit.

$0^\circ \le \theta < 90^\circ$	if satellite travels eastward in its orbit
$\theta = 90^\circ$	if satellite travels due north and south
$90^\circ < \theta \le 180^\circ$	if satellite travels westward in its orbit

ICBM path elevation angle (δ)--the angle between the ICBM's velocity vector and the local horizontal plane.

ICBM path azimuth (λ)--the angle in the local horizontal plane between the projection of the ICBM's velocity vector and the vector pointing due north.

Satellite argument of latitude (ψ)--the (non-negative) central angle in a satellite's orbit plane between the ascending node and the satellite.

Conventions

1. Altitudes are measured from the surface of the earth.
2. $-90^\circ \le \sin^{-1}(\ ) \le 90^\circ$; $0^\circ \le \cos^{-1}(\ ) \le 180^\circ$; $-90^\circ < \tan^{-1}(\ ) < 90^\circ$.
3. Arg(x,y) is the angle (measured counterclockwise) between the positive x-axis and the ray joining (0,0) to (x,y). In particular,

$$\arg(x,y) = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0 \\ \tan^{-1}(y/x) + 180^\circ & \text{if } x < 0 \\ 90^\circ & \text{if } x = 0,\ y > 0 \\ -90^\circ & \text{if } x = 0,\ y < 0 \\ \text{undefined} & \text{if } x = 0,\ y = 0 \end{cases}$$

4. The dot product ($\cdot$) of two vectors $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3)$ is $x \cdot y = x_1y_1 + x_2y_2 + x_3y_3$.
5. The cross product ($\times$) of two vectors $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3)$ is $x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$.
6. The length ($||\ ||$) of a vector $x = (x_1,x_2,x_3)$ is

$$||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

7. Unless otherwise indicated, all variables in the lists and tables that follow are referenced to the rotating xyz coordinate system described earlier.

INPUTS

Constants	Notation	Range
Earth radius (km)	r	6375.58
Earth gravitation constant (km^3/sec^2)	G	398603
Length of sidereal day (sec)	F	86164.1
Earth rotation rate (rad/sec)	ω	7.29212×10^{-5}
Constellation Variables		
Satellite orbit inclination (deg)	θ	$0 \leq \theta \leq 180$
Satellite orbit altitude (km)	h	
Number of rings of satellites	m	
Number of satellites per ring	n	
Ring number of particular satellite	i	
Position number of particular satellite	j	
Constellation phasing fraction	ρ	$0 \leq \rho < 1$
Ring spacing option	ε	$\begin{cases} 1 \text{ if option A} \\ 2 \text{ if option B} \end{cases}$
Longitude of reference satellite at 0-hour (deg)	E	
ICBM Variables		
ICBM launch time (min after 0-hour)	q	
Particular time after ICBM launch (min after 0-hour)	t	$t \geq q$
ICBM altitude at time t (km)	α	
ICBM latitude at time t (deg)	A	
ICBM longitude at time t (deg)	B	
ICBM speed at time t (km/sec)	γ	
ICBM path elevation angle at time t (deg)	δ	
ICBM path azimuth at time t (deg)	λ	

<u>OUTPUTS</u>	<u>Notation</u>
Satellite orbital period (sec)*	s
$$2\pi \sqrt{\frac{(r + h)^3}{G}}$$	
Satellite orbital speed*	
$$\frac{60 \cdot 360}{s} \quad \text{deg/min}$$	k
$$\sqrt{\frac{G}{r + h}} \quad \text{km/sec}$$	z
Angular separation between ring i and ring 1 (deg east from ring 1)	Φ
$$\frac{360(1 - i)}{m\varepsilon}$$	
Longitude of ascending node of ring i at time t (deg)	H
$$\Phi - t\omega \frac{60 \cdot 180}{\pi} + E$$	
Argument of latitude of satellite i-j at time t (deg)	ψ
$$\frac{360(\rho(i - 1) + j - 1)}{n} + kt$$	
Latitude of satellite i-j at time t (deg)	L
$$\sin^{-1}(\sin \psi \sin \theta)$$	
Longitude of satellite i-j at time t (deg)	M
$$H + \arg(\cos \psi, \sin \psi \cos \theta)$$	

---

*With respect to inertial space.

Position vector of satellite i-j at time t (km) X

$$X(1) = (r + h)\cos L \cos M$$
$$X(2) = (r + h)\cos L \sin M$$
$$X(3) = (r + h)\sin L$$

Speed of satellite i-j at time t (km/sec) c

$$\sqrt{z^2 - 2z\omega(r + h)\cos\theta + \omega^2(r + h)^2\cos^2 L}$$

Path azimuth of satellite i-j at time t (deg) b

$$\arg(z\cos\psi\sin\theta,\ z\cos\theta - \omega(r + h)\cos^2 L)$$

Velocity vector of satellite i-j at time t (km/sec) u

$$u(1) = c(-\cos b\sin L\cos M - \sin b\sin M)$$
$$u(2) = c(-\cos b\sin L\sin M + \sin b\cos M)$$
$$u(3) = c\cos b\cos L$$

ICBM position vector at time t (km) Y

$$Y(1) = (r + \alpha)\cos A\cos B$$
$$Y(2) = (r + \alpha)\cos A\sin B$$
$$Y(3) = (r + \alpha)\sin A$$

ICBM velocity vector at time t (km/sec) v

$$v(1) = \gamma(-\cos\lambda\cos\delta\sin A\cos B - \sin\lambda\cos\delta\sin B + \sin\delta\cos A\cos B)$$
$$v(2) = \gamma(-\cos\lambda\cos\delta\sin A\sin B + \sin\lambda\cos\delta\cos B + \sin\delta\cos A\sin B)$$
$$v(3) = \gamma(\cos\lambda\cos\delta\cos A + \sin\delta\sin A)$$

Slant range between satellite i-j and ICBM at time t (km) d

$$\sqrt{(X(1) - Y(1))^2 + (X(2) - Y(2))^2 + (X(3) - Y(3))^2}$$

Closing velocity between satellite i-j and ICBM at time t (km/sec) f

$$\frac{(u - v)\cdot(Y - X)}{d}$$

Angle between ICBM velocity vector at time t and line-of-sight vector between satellite i-j and ICBM (deg) ξ

$$\cos^{-1}\left(\frac{v \cdot (X - Y)}{\gamma d}\right)$$

Minimum altitude along line-of-sight vector joining satellite i-j to ICBM at time t (km) D

$$\begin{cases} \text{Min }\{h,\alpha\} & \text{if } \left[(X - Y) \cdot X\right]\left[(X - Y) \cdot Y\right] \geq 0 \\ \dfrac{\|X \times Y\|}{\|X - Y\|} - r & \text{otherwise} \end{cases}$$

AN EXAMPLE ENGAGEMENT

We will illustrate the use of these formulas by considering a hypothetical engagement in which a 24-satellite constellation is defending against an ICBM launched from 50° latitude, 90° longitude, and with an initial azimuth heading of 10 deg. Trajectory information for such a launch is given in Table 1. We will assume that the launch occurs at 0-hour, so that the five-minute boost phase period shown in the table occurs between times t = 0 and t = 5. Constellation inputs are assumed as follows:

Satellite orbit inclination	θ = 75 deg
Satellite orbit altitude	h = 1250.75 km
Number of rings of satellites	m = 6
Number of satellites per ring	n = 4
Constellation phasing fraction	ρ = 2/3
Ring-spacing option	A (ε = 1)
Longitude of reference satellite at 0-hour	E = 0 deg

We compute

Satellite orbital period	s = 6628.01 sec
Satellite orbital speed	k = 3.26 deg/min
	z = 7.23 km/sec

Table 1

POSITION AND VELOCITY DATA FOR A HYPOTHETICAL ICBM
(Boost Phase)

Launch Information

Launch latitude: 50° N.
Launch longitude: 90° E.
Launch azimuth: 10° (clockwise from north)

Trajectory Data

Time after launch (sec)	Time after 0-hour (min) t	Altitude (km) α	Latitude (deg) A	Longitude (deg) B	Azimuth (deg) λ	Elevation angle (deg) δ	Speed (km/sec) γ	Position vector (km) Y(1)	Y(2)	Y(3)	Velocity vector (km/sec) v(1)	v(2)	v(3)
0.0	0.000	0.0	50.00	90.00	--	90.00	0.00	0.0	4098.1	4884.0	0.00	0.00	0.00
5.0	0.083	0.1	50.00	90.00	9.58	85.23	0.05	-0.0	4098.2	4884.1	-0.00	0.03	0.04
10.0	0.167	0.5	50.00	90.00	9.48	80.77	0.10	-0.0	4098.4	4884.4	-0.00	0.05	0.09
15.0	0.250	1.2	50.00	90.00	9.41	76.73	0.17	-0.0	4098.8	4885.0	-0.01	0.08	0.15
20.0	0.333	2.2	50.00	90.00	9.36	73.07	0.23	-0.1	4099.2	4885.9	-0.01	0.09	0.21
25.0	0.417	3.4	50.01	90.00	9.32	69.77	0.31	-0.1	4099.7	4887.1	-0.02	0.10	0.29
30.0	0.500	5.0	50.01	90.00	9.30	66.79	0.39	-0.2	4100.2	4888.8	-0.02	0.11	0.37
35.0	0.583	6.9	50.02	90.01	9.28	64.09	0.47	-0.4	4100.8	4890.8	-0.03	0.12	0.46
40.0	0.667	9.3	50.03	90.01	9.26	61.65	0.57	-0.6	4101.4	4893.4	-0.04	0.12	0.55
45.0	0.750	12.0	50.05	90.01	9.25	59.46	0.67	-0.8	4102.0	4896.4	-0.06	0.11	0.66
50.0	0.833	15.1	50.06	90.02	9.25	57.49	0.79	-1.1	4102.5	4900.0	-0.07	0.11	0.78
55.0	0.917	18.7	50.08	90.02	9.24	55.73	0.93	-1.5	4103.0	4904.3	-0.08	0.10	0.92
60.0	1.000	22.8	50.11	90.03	9.25	54.15	1.07	-2.0	4103.5	4909.2	-0.10	0.08	1.07
65.0	1.083	27.3	50.14	90.04	9.25	52.75	1.24	-2.5	4103.8	4914.9	-0.12	0.06	1.23
70.0	1.167	32.6	50.17	90.04	9.26	51.51	1.42	-3.2	4104.1	4921.5	-0.14	0.04	1.41
75.0	1.250	38.4	50.22	90.06	9.26	50.40	1.61	-4.0	4104.2	4929.0	-0.17	0.02	1.61
80.0	1.333	45.0	50.27	90.07	9.27	49.42	1.83	-4.8	4104.2	4937.6	-0.19	-0.02	1.82
85.0	1.417	52.4	50.32	90.08	9.29	48.55	2.08	-5.9	4104.1	4947.3	-0.22	-0.05	2.07
90.0	1.500	60.7	50.39	90.10	9.31	47.79	2.36	-7.1	4103.7	4958.3	-0.26	-0.09	2.34
95.0	1.583	70.0	50.46	90.12	9.32	47.13	2.67	-8.4	4103.2	4970.8	-0.29	-0.14	2.65
100.0	1.667	80.3	50.55	90.14	9.35	46.57	3.03	-10.0	4102.3	4984.9	-0.34	-0.19	3.01
105.0	1.750	91.3	50.64	90.16	9.37	46.04	3.05	-11.7	4101.3	5000.0	-0.34	-0.22	3.02
110.0	1.833	102.3	50.73	90.19	9.39	45.52	3.07	-13.5	4100.1	5015.2	-0.35	-0.26	3.04
115.0	1.917	113.2	50.83	90.21	9.42	45.01	3.09	-15.2	4098.7	5030.4	-0.36	-0.29	3.06
120.0	2.000	124.1	50.92	90.24	9.45	44.49	3.12	-17.0	4097.2	5045.7	-0.36	-0.33	3.08
125.0	2.083	135.1	51.02	90.26	9.47	43.98	3.14	-18.9	4095.5	5061.2	-0.37	-0.36	3.10
130.0	2.167	146.0	51.12	90.29	9.50	43.47	3.17	-20.7	4093.6	5076.7	-0.38	-0.40	3.12
135.0	2.250	156.8	51.22	90.32	9.53	42.96	3.19	-22.6	4091.5	5092.3	-0.38	-0.44	3.14

Table 1--continued

Time after launch (sec)	Time after 0-hour (min) t	Altitude (km) α	Latitude (deg) A	Longitude (deg) B	Azimuth (deg) λ	Elevation angle (deg) δ	Speed (km/sec) γ	Position vector (km) Y(1)	Y(2)	Y(3)	Velocity vector (km/sec) v(1)	v(2)	v(3)
140.0	2.333	167.7	51.32	90.34	9.56	42.46	3.22	-24.6	4089.2	5108.1	-0.39	-0.47	3.16
145.0	2.417	178.6	51.42	90.37	9.59	41.96	3.25	-26.6	4086.7	5124.0	-0.40	-0.51	3.19
150.0	2.500	189.5	51.53	90.40	9.63	41.46	3.28	-28.6	4084.1	5140.0	-0.41	-0.55	3.21
155.0	2.583	200.3	51.64	90.43	9.66	40.98	3.32	-30.6	4081.2	5156.1	-0.42	-0.59	3.24
160.0	2.667	211.3	51.74	90.46	9.69	40.49	3.36	-32.7	4078.2	5172.4	-0.42	-0.63	3.27
165.0	2.750	222.1	51.86	90.49	9.73	40.02	3.39	-34.9	4074.9	5188.8	-0.43	-0.67	3.30
170.0	2.833	233.0	51.97	90.52	9.77	39.55	3.43	-37.1	4071.5	5205.3	-0.44	-0.71	3.33
175.0	2.917	244.0	52.08	90.55	9.80	39.09	3.48	-39.3	4067.8	5222.1	-0.45	-0.75	3.36
180.0	3.000	255.0	52.20	90.59	9.84	38.63	3.52	-41.6	4063.9	5239.0	-0.46	-0.80	3.40
185.0	3.083	265.9	52.32	90.62	9.88	38.18	3.57	-43.9	4059.8	5256.0	-0.47	-0.84	3.43
190.0	3.167	277.0	52.44	90.65	9.92	37.74	3.61	-46.3	4055.5	5273.3	-0.48	-0.89	3.47
195.0	3.250	288.0	52.56	90.69	9.97	37.31	3.67	-48.8	4050.9	5290.7	-0.49	-0.93	3.51
200.0	3.333	299.3	52.68	90.73	10.01	36.89	3.72	-51.2	4046.2	5308.4	-0.50	-0.98	3.55
205.0	3.417	310.4	52.81	90.76	10.05	36.48	3.77	-53.8	4041.1	5326.3	-0.52	-1.03	3.59
210.0	3.500	321.6	52.94	90.80	10.10	36.07	3.83	-56.4	4035.8	5344.3	-0.53	-1.08	3.64
215.0	3.583	333.0	53.07	90.84	10.15	35.68	3.90	-59.1	4030.3	5362.7	-0.54	-1.13	3.69
220.0	3.667	344.4	53.20	90.88	10.19	35.30	3.96	-61.8	4024.5	5381.2	-0.55	-1.19	3.74
225.0	3.750	355.8	53.34	90.92	10.24	34.93	4.03	-64.6	4018.4	5400.0	-0.57	-1.24	3.79
230.0	3.833	367.4	53.48	90.96	10.30	34.57	4.10	-67.5	4012.1	5419.1	-0.58	-1.30	3.85
235.0	3.917	379.1	53.62	91.01	10.35	34.21	4.17	-70.4	4005.5	5438.5	-0.60	-1.35	3.90
240.0	4.000	391.0	53.77	91.05	10.40	33.88	4.25	-73.5	3998.6	5458.2	-0.61	-1.41	3.97
245.0	4.083	402.9	53.92	91.10	10.46	33.55	4.34	-76.6	3991.4	5478.2	-0.63	-1.47	4.03
250.0	4.167	414.9	54.07	91.15	10.51	33.24	4.42	-79.7	3983.8	5498.5	-0.64	-1.54	4.10
255.0	4.250	427.1	54.23	91.20	10.57	32.94	4.52	-83.0	3976.0	5519.2	-0.66	-1.60	4.17
260.0	4.333	439.5	54.38	91.25	10.63	32.65	4.61	-86.4	3967.8	5540.2	-0.68	-1.67	4.25
265.0	4.417	452.0	54.55	91.30	10.69	32.38	4.71	-89.8	3959.3	5561.6	-0.70	-1.74	4.33
270.0	4.500	464.7	54.71	91.35	10.76	32.12	4.82	-93.4	3950.4	5583.5	-0.72	-1.81	4.41
275.0	4.583	477.6	54.88	91.41	10.82	31.88	4.94	-97.0	3941.2	5605.7	-0.74	-1.89	4.50
280.0	4.667	490.8	55.06	91.47	10.89	31.65	5.06	-100.8	3931.6	5628.5	-0.76	-1.97	4.60
285.0	4.750	504.2	55.24	91.53	10.96	31.43	5.19	-104.6	3921.5	5651.7	-0.79	-2.05	4.70
290.0	4.833	517.9	55.42	91.59	11.03	31.23	5.32	-108.6	3911.1	5675.5	-0.81	-2.14	4.81
295.0	4.917	531.8	55.61	91.66	11.10	31.05	5.47	-112.8	3900.2	5699.8	-0.84	-2.23	4.93
300.0	5.000	546.1	55.80	91.72	11.18	30.88	5.63	-117.0	3888.8	5724.8	-0.87	-2.32	5.05

The positions, velocities, and azimuths of the 24 satellites at the time of ICBM launch ($t = 0$), shown in Table 2, are found by letting i run from 1 to 6 and j from 1 to 4 in the formulas for ϕ and ψ, and then substituting in the formulas for L, M, X, u, b, and c. For the sake of example, we select satellite 1-2 and show in Table 3 its positions and velocities during the ICBM boost phase. It is constructed like Table 2, except that time is varied between $t = 0$ and $t = 5$, the end of boost phase. Finally, in Table 4, the data of Tables 1 and 3 are combined using the formulas for d, f, ξ, and D to yield the crucial engagement parameters shown.

We see, for example, that the slant range between the ICBM and satellite 1-2 changes very little during the boost phase (thus the relatively low closing velocity). This can be seen geometrically in Fig. 2, which shows the earth shadows of the orbit rings at 0-hour and the ground trace of the ICBM during its boost phase. The fifth column of Table 4 gives the angle between the ICBM velocity vector and the line-of-sight vector between satellite and ICBM. The absolute difference between this angle and 90 deg represents the *incidence angle* (measured from the normal to the ICBM surface) that would be encountered by a laser beam fired from the satellite toward the ICBM* and passing through the ICBM axis.

The last column of Table 4 records the minimum altitude (above the earth's surface) of points along the line-of-sight vector joining the satellite to the ICBM. If this minimum is positive, the ICBM is visible to the satellite above the earth's horizon.

PERIODICITY IN THE ENGAGEMENT GEOMETRY

If the altitude h of the $m \times n$ satellite constellation is chosen properly, the relative geometry between the rotating earth and the constellation repeats every fixed number of days. This phenomenon is an analytic convenience in that it bounds the time period that must be examined in determining the effectiveness of the constellation in defending against ICBM launches. In the paragraphs that follow, we determine the minimum length of such a "repetition period."

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*We assume here that the ICBM is cylindrical with its axis always pointing along its velocity vector.

Table 2

POSITIONS OF SATELLITES IN A 6 × 4 CONSTELLATION

Inclination = 75° Altitude = 1250.75 km Phasing fraction = 2/3

(Time = 0-hour)

Satellite Ring i	Satellite Position j	Latitude (deg) L	Longitude (deg) M	Speed (km/sec) c	Azimuth (deg) b	Position vector (km) X(1)	X(2)	X(3)	Velocity vector (km/sec) u(1)	u(2)	u(3)
1	1	0.00	0.00	7.11	10.66	7626.3	0.0	0.0	0.00	1.32	6.98
1	2	75.00	90.00	7.09	90.00	0.0	1973.8	7366.5	-7.09	0.00	0.00
1	3	-0.00	180.00	7.11	169.34	-7626.3	-0.0	-0.0	0.00	-1.32	-6.98
1	4	-75.00	270.00	7.09	90.00	-0.0	-1973.8	-7366.5	7.09	0.00	-0.00
2	1	56.77	324.15	7.09	26.02	3387.0	-2447.6	6379.5	-2.50	5.64	3.49
2	2	28.88	111.50	7.10	166.56	-2447.6	6213.2	3683.2	-2.76	2.50	-6.05
2	3	-56.77	144.15	7.09	153.98	-3387.0	2447.6	-6379.5	2.50	-5.64	-3.49
2	4	-28.88	291.50	7.10	13.44	2447.6	-6213.2	-3683.3	2.76	-2.50	6.05
3	1	56.77	35.85	7.09	153.98	3387.0	2447.6	6379.5	2.50	5.64	-3.49
3	2	-28.88	68.50	7.10	166.56	2447.6	6213.2	-3683.2	-2.76	-2.50	-6.05
3	3	-56.77	215.85	7.09	26.02	-3387.0	-2447.6	-6379.6	-2.50	-5.64	3.49
3	4	28.88	248.50	7.10	13.44	-2447.6	-6213.2	3683.2	2.76	2.50	6.05
4	1	0.00	0.00	7.11	169.34	7626.3	0.0	0.0	0.00	1.32	-6.98
4	2	-75.00	90.00	7.09	90.00	0.0	1973.8	-7366.5	-7.09	-0.00	-0.00
4	3	-0.00	180.00	7.11	10.66	-7626.3	-0.0	-0.0	-0.00	-1.32	6.98
4	4	75.00	270.00	7.09	90.00	-0.0	-1973.8	7366.5	7.09	0.00	0.00
5	1	-56.77	324.15	7.09	153.98	3387.0	-2447.6	-6379.5	-2.50	5.64	-3.49
5	2	-28.88	111.50	7.10	13.44	-2447.6	6213.2	-3683.3	-2.76	2.50	6.05
5	3	56.77	144.15	7.09	26.02	-3387.0	2447.6	6379.5	2.50	-5.64	3.49
5	4	28.88	291.50	7.10	166.56	2447.6	-6213.2	3683.2	2.76	-2.50	-6.05
6	1	-56.77	35.85	7.09	26.02	3387.0	2447.6	-6379.6	2.50	5.64	3.49
6	2	28.88	68.50	7.10	13.44	2447.6	6213.2	3683.2	-2.76	-2.50	6.05
6	3	56.77	215.85	7.09	153.98	-3387.0	-2447.6	6379.5	-2.50	-5.64	-3.49
6	4	-28.88	248.50	7.10	166.56	-2447.6	-6213.2	-3683.2	2.76	2.50	-6.05

Table 3

POSITION AND VELOCITY OF SATELLITE 1-2 DURING BOOST
PHASE OF HYPOTHETICAL ICBM

Inclination = 75° Altitude = 1250.75 km Phasing fraction = 2/3

6 x 4 constellation ICBM launched at 0-hour

Time after launch (sec)	Time after 0-hour (min) t	Latitude (deg) L	Longitude (deg) M	Speed (km/sec) c	Azimuth (deg) b	Position vector (km) X(1)	X(2)	X(3)	Velocity vector (km/sec) u(1)	u(2)	u(3)
0.0	0.000	75.00	90.00	7.09	90.00	0.0	1973.8	7366.5	-7.09	0.00	0.00
5.0	0.083	75.00	91.03	7.09	91.03	-35.4	1973.8	7366.4	-7.09	-0.00	-0.03
10.0	0.167	74.99	92.06	7.09	92.07	-70.9	1973.8	7366.1	-7.09	-0.01	-0.07
15.0	0.250	74.98	93.08	7.09	93.10	-106.3	1973.8	7365.7	-7.08	-0.01	-0.10
20.0	0.333	74.96	94.11	7.09	94.13	-141.7	1973.7	7365.1	-7.08	-0.01	-0.13
25.0	0.417	74.94	95.13	7.09	95.16	-177.1	1973.6	7364.4	-7.08	-0.02	-0.17
30.0	0.500	74.91	96.15	7.09	96.18	-212.5	1973.5	7363.5	-7.08	-0.02	-0.20
35.0	0.583	74.88	97.16	7.09	97.20	-247.9	1973.4	7362.4	-7.08	-0.03	-0.23
40.0	0.667	74.85	98.17	7.09	98.22	-283.4	1973.3	7361.2	-7.08	-0.03	-0.26
45.0	0.750	74.81	99.18	7.09	99.23	-318.8	1973.1	7359.8	-7.08	-0.03	-0.30
50.0	0.833	74.76	100.18	7.09	100.23	-354.1	1972.9	7358.2	-7.08	-0.04	-0.33
55.0	0.917	74.71	101.17	7.09	101.23	-389.5	1972.7	7356.5	-7.08	-0.04	-0.36
60.0	1.000	74.66	102.16	7.09	102.23	-424.9	1972.5	7354.5	-7.07	-0.04	-0.40
65.0	1.083	74.60	103.14	7.09	103.21	-460.3	1972.3	7352.5	-7.07	-0.05	-0.43
70.0	1.167	74.54	104.11	7.09	104.19	-495.6	1972.1	7350.2	-7.07	-0.05	-0.46
75.0	1.250	74.47	105.07	7.09	105.16	-531.0	1971.8	7347.9	-7.07	-0.05	-0.50
80.0	1.333	74.40	106.03	7.09	106.12	-566.3	1971.5	7345.3	-7.07	-0.06	-0.53
85.0	1.417	74.32	106.97	7.09	107.07	-601.6	1971.2	7342.6	-7.06	-0.06	-0.56
90.0	1.500	74.24	107.91	7.09	108.01	-636.9	1970.9	7339.7	-7.06	-0.07	-0.60
95.0	1.583	74.16	108.84	7.09	108.94	-672.2	1970.5	7336.6	-7.06	-0.07	-0.63
100.0	1.667	74.07	109.75	7.09	109.87	-707.5	1970.2	7333.4	-7.05	-0.07	-0.66
105.0	1.750	73.98	110.66	7.09	110.78	-742.8	1969.8	7330.0	-7.05	-0.08	-0.69
110.0	1.833	73.88	111.56	7.09	111.68	-778.0	1969.4	7326.4	-7.05	-0.08	-0.73
115.0	1.917	73.78	112.44	7.09	112.57	-813.3	1969.0	7322.7	-7.04	-0.08	-0.76
120.0	2.000	73.67	113.32	7.09	113.45	-848.5	1968.6	7318.9	-7.04	-0.09	-0.79
125.0	2.083	73.57	114.18	7.09	114.32	-883.7	1968.1	7314.8	-7.04	-0.09	-0.83
130.0	2.167	73.46	115.03	7.09	115.18	-918.9	1967.7	7310.6	-7.03	-0.09	-0.86
135.0	2.250	73.34	115.87	7.09	116.02	-954.0	1967.2	7306.2	-7.03	-0.10	-0.89

Table 3--continued

Time after launch (sec)	Time after 0-hour (min) t	Latitude (deg) L	Longitude (deg) M	Speed (km/sec) c	Azimuth (deg) b	Position vector (km) X(1)	X(2)	X(3)	Velocity vector (km/sec) u(1)	u(2)	u(3)
140.0	2.333	73.22	116.70	7.09	116.86	-989.1	1966.7	7301.7	-7.02	-0.10	-0.92
145.0	2.417	73.10	117.52	7.09	117.68	-1024.3	1966.2	7297.0	-7.02	-0.11	-0.96
150.0	2.500	72.97	118.32	7.09	118.49	-1059.3	1965.6	7292.1	-7.02	-0.11	-0.99
155.0	2.583	72.85	119.11	7.09	119.29	-1094.4	1965.1	7287.1	-7.01	-0.11	-1.02
160.0	2.667	72.71	119.90	7.09	120.07	-1129.4	1964.5	7281.9	-7.01	-0.12	-1.06
165.0	2.750	72.58	120.67	7.09	120.85	-1164.5	1963.9	7276.5	-7.00	-0.12	-1.09
170.0	2.833	72.44	121.42	7.09	121.61	-1199.5	1963.3	7271.0	-7.00	-0.12	-1.12
175.0	2.917	72.30	122.17	7.09	122.36	-1234.4	1962.7	7265.3	-6.99	-0.13	-1.15
180.0	3.000	72.16	122.90	7.09	123.10	-1269.4	1962.0	7259.5	-6.99	-0.13	-1.19
185.0	3.083	72.01	123.62	7.09	123.83	-1304.3	1961.3	7253.5	-6.98	-0.14	-1.22
190.0	3.167	71.86	124.33	7.09	124.54	-1339.2	1960.7	7247.3	-6.97	-0.14	-1.25
195.0	3.250	71.71	125.03	7.09	125.25	-1374.0	1960.0	7241.0	-6.97	-0.14	-1.28
200.0	3.333	71.55	125.72	7.09	125.94	-1408.8	1959.2	7234.5	-6.96	-0.15	-1.32
205.0	3.417	71.40	126.39	7.09	126.62	-1443.6	1958.5	7227.8	-6.96	-0.15	-1.35
210.0	3.500	71.24	127.06	7.09	127.29	-1478.4	1957.7	7221.0	-6.95	-0.15	-1.38
215.0	3.583	71.07	127.71	7.09	127.95	-1513.1	1957.0	7214.0	-6.94	-0.16	-1.41
220.0	3.667	70.91	128.35	7.09	128.59	-1547.8	1956.2	7206.8	-6.94	-0.16	-1.45
225.0	3.750	70.74	128.98	7.09	129.23	-1582.5	1955.3	7199.5	-6.93	-0.16	-1.48
230.0	3.833	70.57	129.60	7.09	129.85	-1617.1	1954.5	7192.1	-6.92	-0.17	-1.51
235.0	3.917	70.40	130.21	7.09	130.47	-1651.7	1953.7	7184.4	-6.91	-0.17	-1.54
240.0	4.000	70.23	130.81	7.09	131.07	-1686.2	1952.8	7176.6	-6.91	-0.18	-1.58
245.0	4.083	70.05	131.40	7.09	131.66	-1720.8	1951.9	7168.7	-6.90	-0.18	-1.61
250.0	4.167	69.87	131.98	7.09	132.24	-1755.2	1951.0	7160.6	-6.89	-0.18	-1.64
255.0	4.250	69.69	132.54	7.09	132.82	-1789.7	1950.1	7152.3	-6.88	-0.19	-1.67
260.0	4.333	69.51	133.10	7.09	133.38	-1824.1	1949.1	7143.8	-6.88	-0.19	-1.70
265.0	4.417	69.33	133.65	7.09	133.93	-1858.4	1948.2	7135.2	-6.87	-0.19	-1.74
270.0	4.500	69.14	134.19	7.09	134.47	-1892.8	1947.2	7126.5	-6.86	-0.20	-1.77
275.0	4.583	68.95	134.72	7.09	135.01	-1927.0	1946.2	7117.6	-6.85	-0.20	-1.80
280.0	4.667	68.76	135.24	7.09	135.53	-1961.3	1945.2	7108.5	-6.84	-0.21	-1.83
285.0	4.750	68.57	135.75	7.09	136.05	-1995.5	1944.1	7099.2	-6.83	-0.21	-1.86
290.0	4.833	68.38	136.25	7.09	136.55	-2029.6	1943.1	7089.8	-6.83	-0.21	-1.90
295.0	4.917	68.19	136.74	7.09	137.05	-2063.7	1942.0	7080.3	-6.82	-0.22	-1.93
300.0	5.000	67.99	137.22	7.09	137.54	-2097.8	1940.9	7070.6	-6.81	-0.22	-1.96

Table 4

BOOST PHASE ENGAGEMENT PARAMETERS FOR
HYPOTHETICAL ICBM AND SATELLITE 1-2

(ICBM launched at 0-hour)

Time after launch (sec)	Time after 0-hour (min) t	Slant range (km) d	Closing velocity (km/sec) f	Angle between LOS and ICBM direction[a] (deg) ξ	Minimum altitude along LOS vector (km) D
0.0	0.000	3267.3	0.00	---	0.0
5.0	0.083	3267.4	-0.04	77.43	0.1
10.0	0.167	3267.7	-0.07	69.00	0.5
15.0	0.250	3268.1	-0.10	68.55	1.2
20.0	0.333	3268.7	-0.11	63.90	2.2
25 0	0.417	3269.3	-0.11	59.83	3.4
30.0	0.500	3269.9	-0.11	57.46	5.0
35.0	0.583	3270.5	-0.11	54.66	6.9
40.0	0.667	3270.9	-0.10	53.34	9.3
45.0	0.750	3271.3	-0.05	49.94	12.0
50.0	0.833	3271.5	-0.02	48.64	15.1
55.0	0.917	3271.4	0.04	47.02	18.7
60.0	1.000	3271.0	0.12	44.69	22.8
65.0	1.083	3270.3	0.19	43.83	27.3
70.0	1.167	3269.0	0.29	42.76	32.6
75.0	1.250	3267.4	0.40	41.48	38.4
80.0	1.333	3265.1	0.52	40.68	45.0
85.0	1.417	3262.1	0.68	40.02	52.4
90.0	1.500	3258.3	0.85	39.56	60.7
95.0	1.583	3253.6	1.05	38.90	70 0
100.0	1.667	3247.7	1.29	38.41	80.3
105.0	1.750	3241.4	1.25	38.34	91.3
110.0	1.833	3235.1	1.23	37.80	102.3
115.0	1.917	3229.1	1.20	37.47	113.2
120.0	2.000	3223.3	1.16	37.27	124.1
125.0	2.083	3217.5	1.14	36.97	135.1
130.0	2.167	3211.9	1.11	36.72	146.0
135.0	2.250	3206.5	1.08	36.33	156.8

Table 4--continued

Time after launch (sec)	Time after 0-hour (min) t	Slant range (km) d	Closing velocity (km/sec) f	Angle between LOS and ICBM direction (deg) ξ	Minimum altitude along LOS vector (km) D
140.0	2.333	3201.1	1.04	36.32	167.7
145.0	2.417	3195.9	1.02	35.91	178.6
150.0	2.500	3190.9	0.99	35.73	189.5
155.0	2.583	3186.0	0.97	35.61	200.3
160.0	2.667	3181.3	0.94	35.61	211.3
165.0	2.750	3176.6	0.91	35.29	222.1
170.0	2.833	3172.2	0.89	35.24	233.0
175.0	2.917	3167.7	0.85	35.43	244.0
180.0	3.000	3163.5	0.84	35.05	255.0
185.0	3.083	3159.5	0.81	35.29	265.9
190.0	3.167	3155.4	0.79	34.97	277.0
195.0	3.250	3151.5	0.76	35.30	288.0
200.0	3.333	3147.8	0.74	35.26	299.3
205.0	3.417	3144.0	0.72	35.13	310.4
210.0	3.500	3140.5	0.71	35.20	321.6
215.0	3.583	3136.9	0.68	35.50	333.0
220.0	3.667	3133.5	0.67	35.47	344.4
225.0	3.750	3130.3	0.66	35.71	355.8
230.0	3.833	3127.1	0.64	35.79	367.4
235.0	3.917	3123.9	0.62	36.08	379.1
240.0	4.000	3120.7	0.61	36.29	391.0
245.0	4.083	3117.8	0.60	36.72	402.9
250.0	4.167	3114.8	0.59	36.85	414.9
255.0	4.250	3111.9	0.57	37.38	427.1
260.0	4.333	3109.1	0.56	37.52	439.5
265.0	4.417	3106.2	0.57	37.86	452.0
270.0	4.500	3103.5	0.55	38.38	464.7
275.0	4.583	3100.8	0.55	38.86	477.6
280.0	4.667	3098.0	0.55	39.27	490.8
285.0	4.750	3095.3	0.55	39.75	504.2
290.0	4.833	3092.5	0.56	40.18	517.9
295.0	4.917	3089.7	0.56	40.71	531.8
300.0	5.000	3086.9	0.57	41.39	546.1

[a] LOS = line of sight

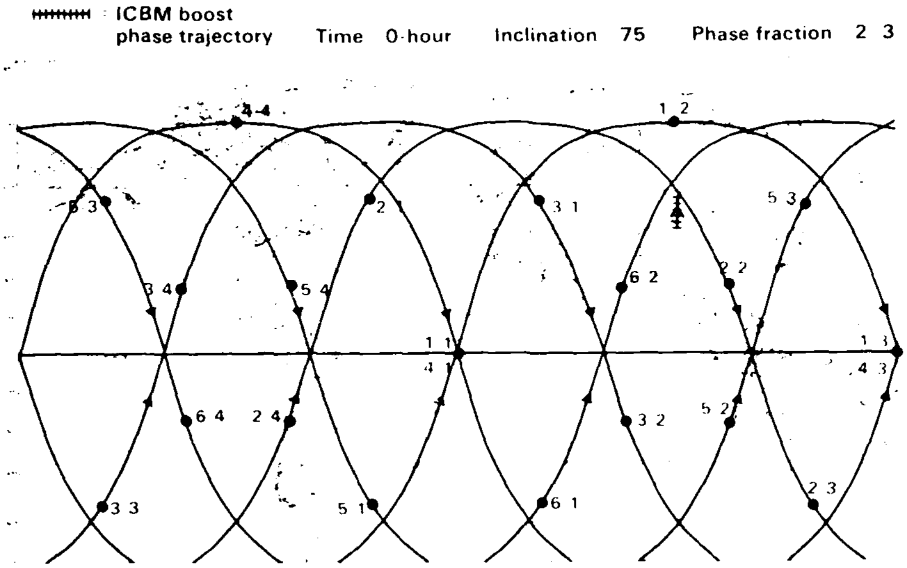

Fig. 2 — Example engagement geometry—6 x 4 constellation

At the altitude h, each satellite makes c orbits per sidereal day, where*

$$c = \frac{86164.1}{s}$$

Imagine a point Q on the earth's surface which at some time 0 is at 0° latitude and 0° longitude. Assume that the reference satellite (1-1) is directly overhead Q at this moment, and heading north. Suppose that p days later, the earth occupies the same position relative to the constellation as it did at time 0. The determination of the minimum possible value for p varies depending on the ring-spacing option assumed (either option A or B).

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*Variables (such as s) that are not defined here are as defined earlier.

Ring-spacing Option A

The point Q, p days after time 0, must again be directly beneath some north-bound satellite. This forces p to be a positive integral multiple of 1/m, the fraction of a day required for the earth to rotate from under one ring to under the one immediately to the east. Thus p = I/m for some positive integer I. The satellites in the Ith ring to the east of ring 1 are shifted ahead of those in ring 1 by $\rho(m - I)/cn$ days. Thus for some satellite in this ring to be at the equator p days after time 0 requires that

$$p = \frac{J}{cn} - \frac{\rho(m - I)}{cn} \quad \text{for some integer } J$$

Equating the two expressions for p gives

$$\frac{I}{m} = \frac{J}{cn} - \frac{\rho(m - I)}{cn}$$

Set $\rho = \ell/m$ (recall that ρ can only assume the values 0, 1/m, 2/m, ..., (m - 1)/m). After substituting and simplifying, we obtain

$$I(cn - \ell) = m(J - \ell)$$

To find the *minimum* period $\hat{p}$ (that is, the smallest possible value for p), it suffices (since p = I/m) to determine the least positive integer I for which this equation has an integer solution J. If c is not a rational number (that is, the quotient of two integers), no solution is possible. If c is rational, express it as the quotient x/y of two integers, in lowest terms. It can be shown that the minimal solution is

$$\hat{I} = \frac{my}{\gcd(nx - \ell y,\ my)}$$

where the greatest common divisor of two integers F and G (gcd(F, G)) is defined as the largest integer that divides evenly into both F and G.

We conclude that the minimal period $\hat{p}$ is given by

$$\hat{p} = \frac{y}{\gcd(nx - \ell y,\ my)} \text{ days}$$

This computation is illustrated below for four different constellation configurations. In example 1, the engagement geometry repeats every day; in example 2, every sixth of a day; in example 3, every three days; and in example 4, every fourth of a day.

Example	m	n	ρ	ℓ	c	h	x	y	nx-ℓy	my	gcd(nx-ℓy,my)	$\hat{p}$
1	6	4	$^1/_2$	3	13	1250.75	13	1	49	6	1	1
2	6	4	$^2/_3$	4	13	1250.75	13	1	48	6	6	$^1/_6$
3	3	8	0	0	$14^1/_3$	770.15	43	3	344	9	1	3
4	4	6	$^1/_2$	2	$14^1/_3$	770.15	43	3	252	12	12	$^1/_4$

Ring-spacing Option B

An m × n constellation configured under this option does not possess the perfect symmetry of one configured under option A. This lack of symmetry results from the existence of a "seam" between rings 1 and m, with satellites in these two rings moving in opposite directions, and those in other pairs of adjacent rings moving in the same direction. As a consequence, repetition periods of less than one day are not possible. If each satellite makes c orbits per day, and $c = x/y$ is a rational number in lowest terms (x and y are integers), then the *minimum* repetition period is $\hat{p} = y/\gcd(y, n)$ days. This formula is illustrated in the table below. Note, in particular, that $\hat{p}$ is independent of the phasing fraction ρ.

Example	m	n	c	h	x	y	gcd(y,n)	$\hat{p}$
1	6	4	13	1250.75	13	1	1	1
2	6	4	$13^1/_2$	1061.26	27	2	2	1
3	8	3	$13^1/_2$	1061.26	27	2	1	2
4	4	6	$13^2/_3$	1000.68	41	3	3	1